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# CALCULATION OF TEMPERATURES AND TEMPERATURE STRESSES IN A STEEL CYLINDRICAL BILLET WITH AN INTERNAL HEAT SOURCE

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*The solution of the problem of heat conduction and thermoelasticity of a cylinder in induction heating has been obtained with the method of equivalent sources. An example of calculation of the temperature and thermal-stress fields for different grades of steel has been given.*

**Introduction.** One of the most progressive methods of high-speed heating of metals before the corresponding pressure treatment (rolling, forging, and stamping) is induction heating [1, 2]. This is particularly true of alloy steels which possess a higher-than-average strain resistance and a high sensitivity to thermal stresses. A comparative analysis of the economic indices of the induction (direct) and flame (indirect) methods of heating unconditionally counts in favor of the former: decarbonization and scaling are reduced several times, the wear of tools (rolls and dies) and rejects decrease, and the yield of high-quality metal increases, which is of particular importance for expensive metals. Therefore, the corresponding theoretical investigations and effective procedures of calculation of the temperature and thermal-stress fields are necessary for improvement of induction plants and regimes of heating of metals.

In certain literature sources (for example, [2, 3]), one notes another positive property of induction heating: direct heating by internal sources distributed throughout the volume of a body removes the problem on temperature stresses. However there is also another viewpoint: "The resulting temperature fields and stresses can attain a considerable value and exceed permissible values" ([4], p. 5).

Thus, the development of procedures of calculation of temperatures and thermal stresses in ingots and billets in induction heating is a topical problem. Below we give the solution of the corresponding problem (of heat conduction and thermoelasticity) in induction heating of cylindrical ingots and billets on the basis of the method of equivalent sources.

**Formulation and Solution of the Problem.** Let us consider a long continuous cylinder of diameter $2R$ with an initial temperature of $T_0$ = const. In view of the very short duration of the initial stage of warming up, we will consider the steady-state regime of heating of the metal. The heat loss to the ambient medium is disregarded, i.e., the cylinder is considered to be heat-insulated.

In this case, the corresponding boundary-value problem of heat conduction has the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta}{\partial\rho}\right) + \text{Po}\exp\left[-2\left(1-\rho\right)\right] = \frac{\partial\theta}{\partial\tau}\,, \qquad (1)$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = \left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=1} = 0\,, \quad \left.\theta\left(\rho, \tau\right)\right|_{\substack{\tau=\tau_0\\ \rho=0}} = \theta_c\left(\tau_0\right) = 0\,, \qquad (2)$$

where $\rho = \dfrac{r}{R}$, $\theta(\rho, \tau) = \dfrac{T(r, t) - T_0}{T_{cr} - T_0}$, $\text{Po} = \dfrac{q_s R}{\lambda(T_f - T_0)}$, $\tau = \dfrac{at}{R^2}$, $\tau_0 = \dfrac{a_0 t}{R^2}$, and $\theta_c = \dfrac{T_c - T_0}{T_{cr} - t_0}$.

Taking the solution of problem (1), (2) as the "load" function $\theta(\rho, \tau)$, we determine the thermally stressed state of the cylinder by the well-known (for example, [5–9]) solution of the corresponding quasistatic unconnected

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thermoelasticity problem determining three components of the stress tensor: the radial ($\sigma_r$), tangential ($\sigma_\varphi$), and axial ($\sigma_z$) components:

$$\bar{\sigma}_r(\rho, \tau) = \int_0^1 \theta(\rho, \tau)\, \rho d\rho - \frac{1}{\rho^2} \int_0^\rho \theta(\rho, \tau)\, \rho d\rho\,, \tag{3}$$

$$\bar{\sigma}_\varphi(\rho, \tau) = \int_0^1 \theta(\rho, \tau)\, \rho d\rho + \frac{1}{\rho^2} \int_0^\rho \theta(\rho, \tau)\, \rho d\rho - \theta(\rho, \tau)\,, \tag{4}$$

$$\bar{\sigma}_z(\rho, \tau) = 2 \int_0^1 \theta(\rho, \tau)\, \rho d\rho - \theta(\rho, \tau)\,. \tag{5}$$

The analytical solution of the boundary-value problem of heat conduction (1), (2) is known [10]. However, it is represented by an unbounded series containing Bessel functions and exponential time functions that necessitate solution of transcendental equations for determination of characteristic numbers. The structural complexity of such a solution presents certain practical difficulties in calculations of the temperature and particularly the thermally stressed state of the cylinder. Meanwhile, different approximate analytical methods yielding solutions very simple in form and quite exact as far as calculation results are concerned are widely used in applied heat engineering. Among these methods is the method of equivalent sources having manifested itself well in solution of a wide class of linear and nonlinear problems of heat conduction and thermomechanics [6, 7], including the cases with allowance for the action of continuously distributed internal heat sources [11]. Let us apply this method to problem (1), (2). We take the resolvent of the method of equivalent sources in the form

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \theta}{\partial \rho} \right) = f(\tau) - \mathrm{Po} \exp\left(-2(1-\rho)\right)\,, \tag{6}$$

where the "equivalent source" $f(\tau)$ is determined by the integral condition

$$f(\tau) = \frac{d}{d\tau} \int_0^t \theta(\rho, \tau)\, d\rho\,. \tag{7}$$

Integrating Eq. (6) with respect to $\rho$, we obtain

$$\frac{\partial \theta}{\partial \rho} = \frac{f(\tau)\rho}{2} - \frac{\mathrm{Po}}{4} \exp\left(-2(1-\rho)\right) \left(2 - \frac{1}{\rho}\right) + \frac{A(\tau)}{\rho}\,. \tag{8}$$

Substituting expression (8) into boundary conditions (2), we find

$$A(\tau) = -\frac{\mathrm{Po}}{4} \exp(-2)\,, \quad f(\tau) = \frac{\mathrm{Po}}{2} \left(1 + \exp(-2)\right)\,, \tag{9}$$

after which the heat flux is determined by the function

$$\frac{\partial \theta}{\partial \rho} = \frac{\mathrm{Po}}{4} \left[ \left(1 + \exp(-2)\right) \rho - \exp\left(-2(1-\rho)\right) \right] \left(2 - \frac{1}{\rho}\right) + \frac{A(\tau)}{\rho}\,. \tag{10}$$

Integrating expression (10) with respect to $\rho$, we obtain

$$\theta(\rho, \tau) = B(\tau) + \frac{\text{Po}}{4}\left\{(1 + \exp(-2))\frac{\rho^2}{2} - \exp(-2)\left[\exp(2\rho) - \Phi(\rho)\right]\right\}, \tag{11}$$

where we have introduced the function

$$\Phi(\rho) = \int_0^\rho \varphi(\rho')\, d\rho'; \quad \varphi(\rho') = \frac{\exp(2\rho') - 1}{\rho'}. \tag{12}$$

It is clear that the arbitrary function of integration $B(\tau)$ is related to the temperature of the center of the cylinder cross section by the expression $B(\tau) = \theta_c(\tau) + \text{Po}\,\frac{\exp(-2)}{4}$. This enables us to write the solution in the following form:

$$\theta(\rho, \tau) = \theta_c(\tau) + \frac{\text{Po}}{4}\left\{(1 + \exp(-2))\frac{\rho^2}{2} + \exp(-2)\left[1 - \exp(2\rho) + \Phi(\rho)\right]\right\}. \tag{13}$$

Substituting the expressions of $f(\tau)$ from (9) and $\theta(\rho, \tau)$ from (13) into integral condition (7), we obtain the differential equation $2d\theta_c(\tau) = \text{Po}\,(1 + \exp(-2))d\tau$ after whose integration, with account for initial ($\tau = \tau_0$) condition (2), we determine the temperature function of the center of the cylinder cross section

$$\theta_c(\tau) = \frac{\text{Po}}{2}(1 + \exp(-2))(\tau - \tau_0) \tag{14}$$

and the final solution of the thermal problem

$$\theta(\rho, \tau) = \frac{\text{Po}}{4}\left\{(1 + \exp(-2))\left[2(\tau - \tau_0)\right] + \exp(-2)\left[1 - \exp(-\rho) + \Phi(\rho)\right]\right\}. \tag{15}$$

By direct substitution of expression (15) into the initial mathematical model we easily assure ourselves that the approximate solution obtained exactly satisfies the differential heat-conduction equation (1) and all boundary conditions (2). The degree of approximation of the solution (15) is that it does not cover the initial stage ($0 \le \tau \le \tau_0$) of warming up specified by the classical Fourier theory; by the end of this stage, there forms the initial (for the steady-state regime $\tau \ge \tau_0$) temperature field described, according to our solution, by the function

$$\theta(\rho, \tau_0) = \theta_0(\rho) = \frac{\text{Po}\exp(-2)}{4}\left[\frac{1 + \exp(2)}{2}\rho^2 + 1 - \exp(2\rho) + \Phi(\rho)\right]. \tag{16}$$

This period is very brief and it is usually disregarded [2, 3]. As far as the solution (15) is concerned, the numerical experiment has shown a fairly high accuracy of the results obtained with the method of equivalent sources in many problems, including those with allowance for internal heat sources [9].

It is common knowledge that between the radial, tangential, and axial stresses in a free cylinder, there is the relationship $\sigma_z = \sigma_r + \sigma_\varphi$ for which, for the continuous cylinder, we have $\sigma_r(1) = 0$ and $\sigma_\varphi(1) = \sigma_z$ on the surface ($\rho = 1$) and $\sigma_r(0) = \sigma_\varphi(0) = \sigma_z(0)/2$ at the center of the cross section ($\rho = 0$). On this basis, for investigation of the thermally stressed state of the cylinder it is sufficient to determine the axial stresses. We emphasize that the solution (15) contains the integral of $\Phi(\rho)$ (12) that is not taken in quadratures. This presents certain difficulties when one seeks to obtain the analytical solution. The analysis of the integrand $\varphi(\rho)$ (12) shows that it represents a smooth curve varying within the limits $2 \le \varphi(\rho) \le (\exp(2) - 1)$.

Let us approximate the function $\varphi(\rho)$ (12) by the parabola $\overline{\varphi}(\rho) = b + c\rho + d\rho^2$ whose values, at three points, coincide with the values of the functions $\varphi(\rho)$: $\overline{\varphi}(0) = \varphi(0) = 2$, $\overline{\varphi}(0,5) = \varphi(0,5) = 2(\exp - 1)$, and $\overline{\varphi}(1) = \varphi(1) = \exp(2) - 1$. Solving the corresponding system of three algebraic equations, we find the necessary coefficients $b$, $c$, and $d$ and then the function

TABLE 1. Thermophysical Characteristics of St.10 and KhI8N9T Steels [4]

| Billet No. | Steel grade | $\lambda$, W/(m·K) | $\alpha_T \cdot 10^5$, 1/K | $E \cdot 10^5$, MPa | $\nu$ |
|---|---|---|---|---|---|
| 1 | St.10 | 73.3 | 1.16 | 2.11 | 0.28 |
| 2 | KhI8N9T | 16.7 | 1.70 | 1.92 | 0.253 |

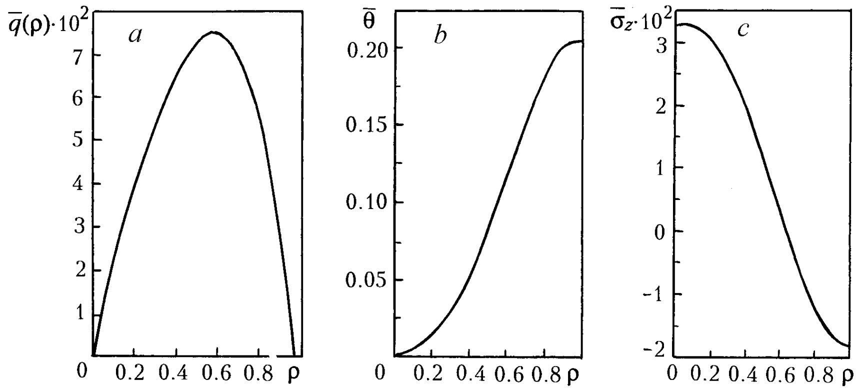


Fig. 1. Distributions of the relative heat flux $\overline{q}(\rho)$ (a), the function of temperature $\theta(\rho)$ (b), and the function of axial stresses $\overline{\sigma}_z(\rho)$ (c) along the cylinder radius.

$$\overline{\varphi}(\rho) = 2 + 1.359\rho + 3.030\rho^2 . \tag{17}$$

Since the function $\overline{\varphi}(\rho)$ from (17) is involved in the solution (15) in the integral form $\Phi$ (12), the error of the performed approximation of (17) will be somewhat smoothed.

In such a case the function $\theta(\rho, \tau)$ of (15) takes the form

$$\theta(\rho, \tau) = \frac{\text{Po}}{4}\left\{(1 + \exp(2))\left[2(\tau - \tau_0) + \frac{\rho^2}{2}\right] + \exp(-2)(1 - \exp(2\rho) + 2\rho + 0.68\rho^2 + 1.01\rho^3)\right\} . \tag{18}$$

We substitute the "load" function of (18) into the solution (15) of the problem of thermoelasticity of the cylinder. After simple transformation we have

$$\overline{\sigma}_z(\rho) = \frac{\text{Po}}{4}\left[\left(\frac{1 + \exp(-2)}{4}\right)(1 + 2\rho^2) - \exp(2(1 - \rho)) - (2.078 - 2\rho - 0.68\rho^2 - 1.01\rho^3)\exp(-2)\right] . \tag{19}$$

Expressions (15) and (19) completely determine the temperature and thermally stressed states of the cylinder under the conditions of steady-state heating.

With the use of the solution obtained for the problem of heat conduction and thermoelasticity we carry out calculations of cylindrical billets of radius $R$ = 0.2 m for two grades of steel: St.10 low-carbon steel ((1)) and KhI8N9T heat-resistant steel ((2)) whose necessary characteristics are given in Table 1. The surface heat flux will be taken to be $q_s = 5 \cdot 10^5$ W/m$^2$ [2].

Figure 1 shows the fields of the relative values of the heat flux $\overline{q}(\rho) = q(\rho)/q_s$, the temperature $\overline{\theta}(\rho) = 4[\theta(\rho) - \theta_c]/\text{Po}$, and the axial stress $\overline{\sigma}_z = \sigma/(K_\sigma \text{Po})$.

Passing to the absolute values of the stresses, we compute the necessary numbers from the data of Table 1:

$$\text{Po}_1 = \frac{q_s R}{\lambda_1 T_f} = \frac{5 \cdot 10^5 \cdot 0.2}{77.3 \cdot T_f} = \frac{1.30 \cdot 10^3}{T_f} , \quad \text{Po}_2 = \frac{5 \cdot 10^5 \cdot 0.2}{16.7 \cdot T_f} = \frac{6.0 \cdot 10^3}{T_f} ,$$

$$K_{\sigma_1} = \frac{\alpha_{T_1} E_1 T_f}{1 - \nu_1} = \frac{1.16 \cdot 10^{-5} \cdot 2.11 \cdot 10^5}{1 - 0.28} T_f = 3.40 T_f , \quad K_{\sigma_2} = \frac{1.7 \cdot 10^{-5} \cdot 1.92 \cdot 10^5 T_f}{1 - 0.253} = 4.37 T_f .$$

Employing the data obtained and Fig. 1c, we find

$$\sigma_{z_{max}}^{(1)} = 3.32 \cdot 10^{-2} \cdot 3.4 \cdot 1.3 \cdot 10^{3} = 147 \text{ MPa},$$

$$\sigma_{z_{max}}^{(2)} = 3.32 \cdot 10^{-2} \cdot 4.37 \cdot 6 \cdot 10^{3} = 870 \text{ MPa}.$$

It is clear that the values of the maximum thermal stresses developing in the cylindrical billet in induction heating substantially depend on the thermophysical and physicomechanical properties of the steel.

**Conclusions.** We have obtained the solution of the problem of thermomechanics (in induction heating); the approximate solution of the heat-conduction problem, which is based on the use of the method of equivalent sources, has been employed. The results of calculation of temperatures, thermal stresses, and heat fluxes in heating of cylindrical billets have shown an accuracy sufficient for engineering practice. The solution obtained for the problem of heat conduction and thermomechanics can be recommended for the calculation practice of induction heating of metal.

## NOTATION

$a$, thermal diffusivity of the body; $E$, elastic modulus; $K_\sigma$, coefficient of conversion of dimensionless stresses to dimensional stresses; Po, Pomerantsev number; $q_s$, surface density of the heat flux; $R$, cylinder radius; $T(r, t)$, $T_0$, and $T_f$, running, initial, and final temperature; $t$, time; $r$, absolute coordinate; $\alpha_T$, coefficient of linear expansion of the body; $\lambda$, thermal conductivity of the body; $\nu$, Poisson coefficient; $\theta$ and $\theta_c$, dimensionless running temperature and temperature of the center; $\rho$, dimensionless coordinate; $\tau_0$ and $\tau$, dimensionless initial and running time; $\sigma$, running stress; $\bar{\sigma}_r$, $\bar{\sigma}_\varphi$, and $\bar{\sigma}_z$, dimensionless radial, tangential, and axial components of the stress tensor. Subscripts: f, final value; cr, crystallization; s, surface; c, center; 0, initial value; max, maximum.

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